NP-Completeness

Class of P Problems:

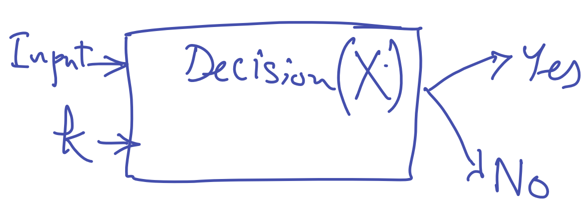
↳ The set of Problems for which there exist a Polynomial-Time Solution

Different Versions of a Problem

Optimization Version: when the goal is to max/min an objective value

Decision Version:

given a target value k, if theres exists a Solution for the Problem P, Such that the objective value $\leq k$ (Min) / $\geq k$ (Max)



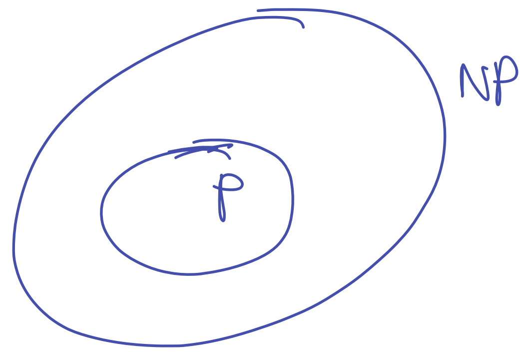
- Verification Version:

Given a Certificate (a potential Solution), V the goal is to verify if V is a Valid Solution to the Problem (Decision Version).

NP (Non-deterministic Polynomial)

A problem $X \in NP$ if the verification Verison of $X \in P$.

$$P \subseteq NP$$



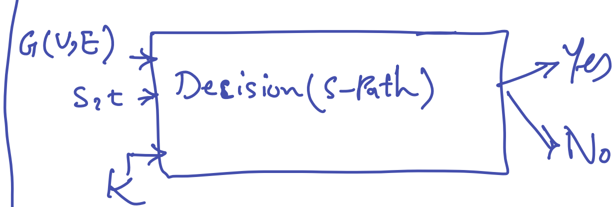
① $P = NP$

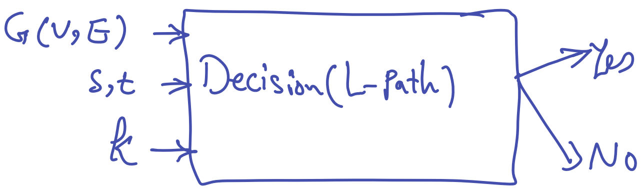
② $P \neq NP$

? UNKNOWN

Shortest Path: ✓ (S-Path)

(L-Path) Longest Path: given s, t, what is the longest Simple Path $s \rightsquigarrow t$





Q_1: S-Path $\in P$?

Yes (Dijkstra)

Q_2: S-path $\in NP$?

Yes $\forall X \in P \Rightarrow X \in NP$

Q_3 = L-Path $\in P$?

Q_4 = L-Path $\in NP$?

Verification(L-Path) $\in P$

Given $G(V,E)$, s, t, k, and a Certificate (C), verify if C is a path from s to t with Length at least k!

Solution:

- C should start with s & end in t
- $\forall$ edge $e \in C$, $e \in E$
- C should be a chain (valid path)
- C ~ contain at least k edges

$O(m)$ $O(n)$ ✓$O(k)$ $O(mn)$

$\Rightarrow$ L-Path $\in NP$

NP-Complete:

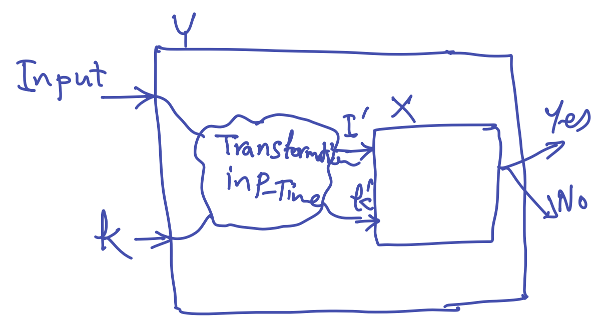
a problem $X \in$ NP-Complete if

① $X \in NP$

② $\forall Y \in NP$, $Y \leq_p X$

In P-time, Y can be Translated to X; such that a Yes/No to X, gives Yes/No to Y.

$(Y \leq_p X)$: Y is Reduced to X in P-time



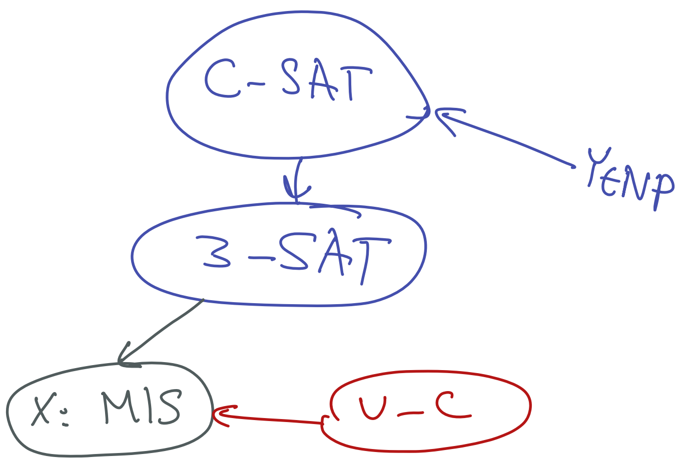
Circuit SAT (Satisfiability):

given boolean variables $v_1 \ldots v_n$ and a boolean expression over the variables, does it exist an assignment to $v_1 \ldots v_n$ s.t. the expression becomes true?

Circuit-SAT $\in$ NP-Complete

3-SAT: given a set of variables $U_1, \dots, U_n$ and a set clauses $C_1 \dots C_m$,
such that C_i contains exactly 3 variables (or their neg)
C_i contains only ($\vee$) OR operation
the expression is
$$C_1 \wedge C_2 \cdots \wedge C_m$$
, is there an assignment to $U_1 \dots U_n$ s.t $C_1 \cdots C_m = $ True

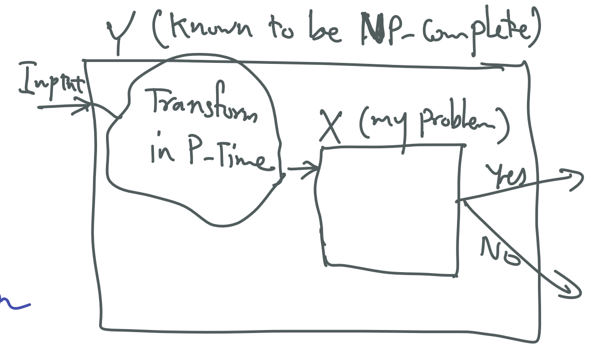
e.g.
$$\underbrace{(V_1 \vee \bar{V}_2 \vee V_3)}_{C_1} \wedge \underbrace{(V_4 \vee V_2 \vee V_3)}_{C}$$
$V_1 = $ True, $V_4 = $ True ✓



Step1: $X \in NP$: A certificate can be verified in P-time

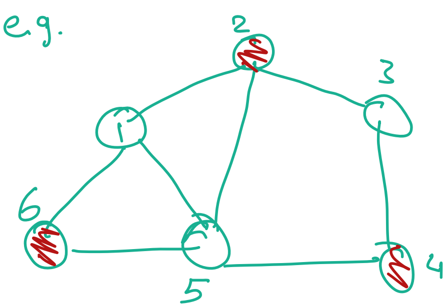
Step2: $\exists\, Y \in$ NP-Complete
$$Y \le_p X$$

Step2:



Maximum Indep. Set (MIS)
Given a graph $G(V,E)$, find the max # of nodes that are indep.
$$\forall \{u,v\} \in S \subseteq V$$
$$(u,v) \notin E$$

e.g.



$\{6\}$ is Indep. Set

$\{6,4,3\}$ X
$(4,3) \in E$

Max indep set? 3
$\{6,2,4\}$

MIS $\in$ NP-Complete

Step1: MIS $\in$ NP ✓

Given $G(V,E)$, a value k, a Certificate S, we can verify S is an Indep. Set of Size k:

if $|S| < k$ return false

$\forall \{u,v\} \in S$

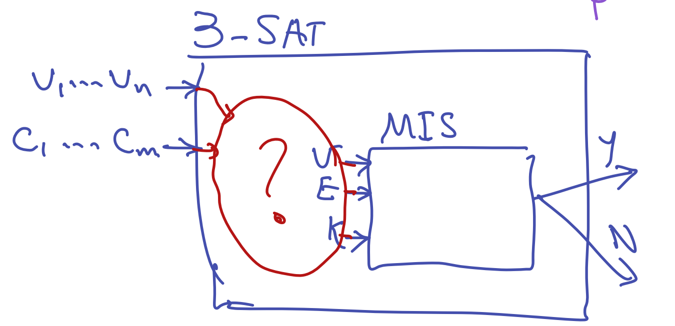
if $(u,v) \in E$

return false

Return True

$O(n^2)$

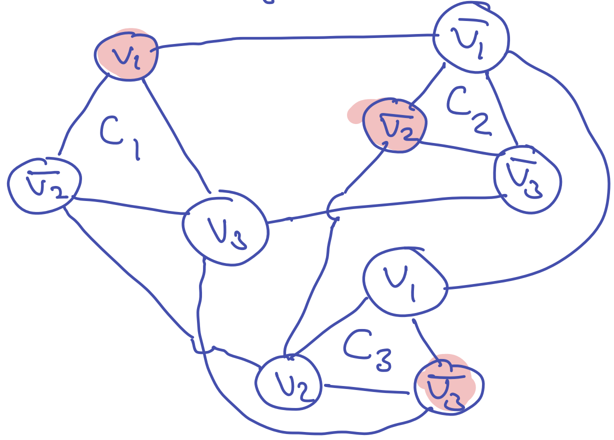
Step 2: Reduction: $3SAT \leq_p MIS$

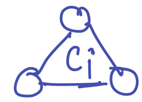


$C_1: (v_1 \vee \bar{v}_2 \vee v_3) \wedge$

$C_2: (\bar{v}_1 \vee \bar{v}_2 \vee \bar{v}_3) \wedge$

$C_3: (v_1 \vee v_2 \vee \bar{v}_3)$



$\forall C_i$ add the gadget 

$\forall v_j \in C_i$, add an edge to every $\bar{v}_j \in C_k$

$k = m$

$\Rightarrow$ The answer to 3SAT is True iff the answer to Decision(MIS) is True

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Vertex-Cover:

Given $G(V,E)$, find the minimum # of vertices such that

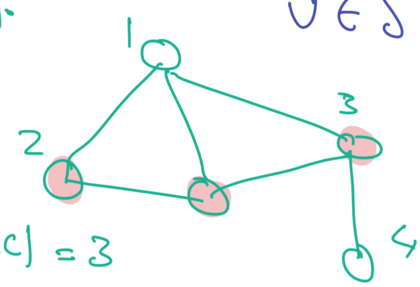
$\forall (u,v) \in E$

$u \in S$

OR

$v \in S$

e.g.



$|V\text{-}C| = 3$

V-C $\in$ NP-Complete

Step1: V-C $\in$ NP ✓

Step2: Reduction

$MIS \leq_p V\text{-}C$